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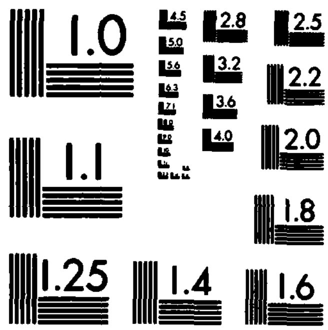

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ABSTRACT

The consistent directions of the least squares estimators in a linear model are defined to be the linear combinations of parameter estimates that are asymptotically consistent. When the design variable is univariate and the regression function is smooth, consistent directions are characterized in previous papers (Wu, 1980; Wu and Wang, 1982) in terms of the convergence rates of the design sequence to its limit points. Extensions of these results to multivariate design variables are considered in the present paper.

AMS (MOS) Subject Classifications: Primary 62J05, Secondary 62E20

Key Words: Asymptotic consistency, Consistent direction, Least squares estimators, Multiple linear regression, Multiple polynomial regression

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SIGNIFICANCE AND EXPLANATION

A basic requirement of the least squares estimator in a linear model is that it should be close to the true parameter for a large sample size. If this is not the case, one would like to know what linear combinations of the components of the least squares estimator are close to their counterparts in the parameter. In this paper we study the characterization of these combinations for linear models with smooth regression functions and multivariate input variables.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

CONSISTENT DIRECTIONS OF THE LEAST SQUARES ESTIMATORS IN LINEAR MODELS

Song-Gui Wang* and C. F. Jeff Wu

1. Introduction

Consider a linear model

$$y = \theta' f(x) + \varepsilon \tag{1.1}$$

where $\theta' = (\theta_1,\ldots,\theta_p)$, $f'(x) = (f_1(x),\ldots,f_p(x))$, $f_j(x)$, $j = 1,\ldots,p$, are univariate functions of the input vector $x = (x_1,\ldots,x_q)'$, the random error ε has mean zero, variance σ^2 and errors corresponding to different observations are uncorrelated or independent. If y_i is observed at x_i, $i = 1,\ldots,n$, and $X_n' = [f(x_1),\ldots,f(x_n)]$ is of full rank, the least squares estimator (LSE) of θ is

$$\hat{\theta} = (X_n'X_n)^{-1}X_n'y\ , \tag{1.2}$$

where $y' = (y_1,\ldots,y_n)$. It is known that $\hat{\theta} \to \theta$ a.s. (or in prob.) iff $(X_n'X_n)^{-1} \to 0$ when $\{\varepsilon_i\}_1^n$ are i.i.d. (or uncorrelated). The strong consistency part was proved in Lai et al. (1979). For recent results on the consistency of LSE, see the references of Wu (1980) and Wu and Wang (1982). In case $(X_n'X_n)^{-1} \to 0$ does not hold, $\hat{\theta}$ is not consistent for estimating the vector θ. It was observed in Wu (1980) that, in this case, the best linear unbiased estimator $b'\hat{\theta}$ of a linear combination $b'\theta = \sum_1^p b_i\theta_i$ may still be consistent for some vectors b. Such a vector b is called a consistent direction of the LSE $\hat{\theta}$. The space of consistent directions is defined as

$$S(f) = \left\{b : b'(X_n'X_n)^{-1}b = b'\left(\sum_1^n f(x_i)f(x_i)'\right)^{-1}b \to 0 \quad \text{as} \quad n \to \infty\right\} .$$

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A general characterization of $S(\underset{\sim}{f})$ as given in Wu (1980). When the input x is scalar ($q = 1$ in (1.1)) and f_j is any smooth function in x, a more refined characterization was given in Wu and Wang (1982) in terms of the convergence rates of the design sequence to its limit points. More specifically, if a is a limit point, i.e., there exists a subsequence $x_{n_i} \to a$ as $i \to \infty$, and $\sum_1^\infty (x_{n_i} - a)^{2r} = \infty$, the r^{th} derivative $\underset{\sim}{f}^{(r)}(a)$ is a consistent direction under some smoothness assumption on $\underset{\sim}{f}(x)$ near a. Extension of this result to multivariate input $\underset{\sim}{x}$ is nontrivial because different components of the vectors $\underset{\sim}{x}_{n_i}$ may converge to their limit points at different rates. A simple special case (constant convergence rate for all components) was considered in Wu and Wang (1982). The purpose of this paper is to present more general results for the multivariate input variable $\underset{\sim}{x}$.

The mathematical results developed in the three papers may find applications beyond the consistency of LSE. For the convenience of the potential users of our results, we will give this problem an equivalent formulation that is void of statistical jargons.

A. Given a sequence of $p \times 1$ vectors $\{\underset{\sim}{v}_i\}_1^\infty$, characterize the subspace $S = \{\underset{\sim}{b} : \underset{\sim}{b}'(\sum_1^n \underset{\sim}{v}_i\underset{\sim}{v}_i')^{-1}\underset{\sim}{b} \to 0 \text{ as } n \to \infty\}$ in terms of the "limiting" behavior of $\{\underset{\sim}{v}_i\}_1^\infty$.

B. If $\underset{\sim}{v}_i$ is a smooth function of a $q \times 1$ vector $\underset{\sim}{w}_i$, i.e. $\underset{\sim}{v}_i = \underset{\sim}{f}(\underset{\sim}{w}_i)$ and $\underset{\sim}{f}$ is smooth, characterize the subspace S defined in A in terms of the "limiting" behavior of the input sequence $\{\underset{\sim}{w}_i\}_1^\infty$. Typically q is much smaller than p.

Problem A was solved in Wu (1980). Problem B was solved for some $\underset{\sim}{f}$ functions and input variables $\underset{\sim}{w}$ in Wu (1980), Wu and Wang (1982) and the present paper.

2. Main results

Before stating the main results, we define some notations. We assume that the input sequence $\{\underset{\sim}{x}_i\}_1^\infty$ is bounded, where $\underset{\sim}{x}_i' = (x_{i1},\dots,x_{iq})$. The less interesting case that $\{\underset{\sim}{x}_i\}_1^\infty$ is unbounded can be treated as in Wu (1980). If there exists an infinite subsequence n_i such that $\underset{\sim}{x}_{n_i} \to \underset{\sim}{a}$ as $i \to \infty$, we say that $\underset{\sim}{a}$ is a limit point of $\{\underset{\sim}{x}_i\}_1^\infty$. Define $N(\underset{\sim}{a},\delta)$, the δ-neighborhood of $\underset{\sim}{a}' = (a_1,\dots,a_q)$, to be $\{\underset{\sim}{x}' = (x_1,\dots,x_q) : |x_i - a_i| < \delta,\ i = 1,\dots,q\}$, and $C^r(\underset{\sim}{a},\delta)$ to be the set of functions with r^{th} continuous partial derivatives in $N(\underset{\sim}{a},\delta)$. In particular, $C^0(\underset{\sim}{a},\delta)$ is the set of continuous functions in $N(\underset{\sim}{a},\delta)$. Define $\underset{\sim}{f}(\underset{\sim}{x}) \in C^r(\underset{\sim}{a},\delta)$ iff $f_i(\underset{\sim}{x}) \in C^r(\underset{\sim}{a},\delta)$ for $1 \le i \le p$. The little o-notation $u_n = o(v_n)$ means $|u_n|/|v_n| \to 0$ as $n \to \infty$.

Define $(\underset{\sim}{x} - \underset{\sim}{a})\nabla = \sum_{i=1}^{q} (x_i - a_i)\dfrac{\partial}{\partial x_i}$,

$$[(\underset{\sim}{x} - \underset{\sim}{a})\nabla]^k \underset{\sim}{f}(\underset{\sim}{a}) = ([(\underset{\sim}{x} - \underset{\sim}{a})\nabla]^k f_1(\underset{\sim}{a}),\dots,[(\underset{\sim}{x} - \underset{\sim}{a})\nabla]^k f_p(\underset{\sim}{a}))'$$

$$= \sum_{\underset{\sim}{t}_k \in T_k} (x_{j_1} - a_{j_1}) \cdots (x_{j_k} - a_{j_k}) \frac{\partial^k \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_{j_1} \cdots \partial x_{j_k}}, \qquad (2.1)$$

$$S_i(k,t) = \frac{\underset{\sim}{w}'[(\underset{\sim}{x}_{n_i} - \underset{\sim}{a})\nabla]^k \underset{\sim}{f}(\underset{\sim}{a})}{k!(x_{n_i t} - a_t)^r} \qquad (2.2)$$

and

$$S_i(r,t,A) = \underset{\sim}{w}' \sum_{\underset{\sim}{t}_r \in A} (x_{n_i j_1} - a_{j_1}) \cdots (x_{n_i j_r} - a_{j_r}) \frac{\partial^r \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_{j_1} \cdots \partial x_{j_r}} \Big/ [r!(x_{n_i t} - a_t)^r], \qquad (2.3)$$

where $\underset{\sim}{t}_k = (j_1,\dots,j_k)$,

$$T_k = \{\underset{\sim}{t}_k = (j_1,\dots,j_k) : 1 \le j_m \le q,\ 1 \le m \le k\}, \qquad (2.4)$$

and A is a subset of T_r, $k = 1,\dots,r$.

Suppose $\{J_\ell\}_{\ell=1}^3$ is a partition of $J = \{1,2,\dots,q\}$. Define a partition of T_r as follows:

$$T_{r1} = \{\underset{\sim}{t}_r = (j_1,\dots,j_r) \in T_r\colon j_m \in J_1 \cup J_2 \text{ for all } m \text{ and } j_m \in J_1 \text{ for at least one } m\}\,, \tag{2.5}$$

$$T_{r2} = \{\underset{\sim}{t}_r = (j_1,\dots,j_r) \in T_r\colon j_m \in J_2 \text{ for all } m\}\,, \tag{2.6}$$

$$T_{r3} = \{\underset{\sim}{t}_r = (j_1,\dots,j_r) \in T_r\colon j_m \in J_3 \text{ for at least one } m\}\,. \tag{2.7}$$

We are now in the position to state the main theorems.

Theorem 1. If $\underset{\sim}{f}(\underset{\sim}{x}) \in C^0(\underset{\sim}{a},\delta)$ for some $\delta > 0$, then $\underset{\sim}{f}(\underset{\sim}{a})$ is a consistent direction for any limit point $\underset{\sim}{a}$ of the design sequence $\{\underset{\sim}{x}_i\}_{i=1}^{\infty}$.

This follows immediately from Theorem 2 of Wu (1980).

Theorem 2. Suppose

(i) $\underset{\sim}{x}_{n_i} \to \underset{\sim}{a}$ as $i \to \infty$, where $\underset{\sim}{x}'_{n_i} = (x_{n_i1}, x_{n_i2},\dots,x_{n_iq})$,

(ii) $\underset{\sim}{f}(\underset{\sim}{x}) \in C^{r_0+1}(\underset{\sim}{a},\delta)$ for some $\delta > 0$, r_0 integer,

(iii) let $\{J_\ell\}_{\ell=1}^{3}$ be a partition of the set $J = \{1,2,\dots,q\}$ satisfying the following conditions for a fixed $j_0 \in J_2$

$$\lim_{i\to\infty} \frac{x_{n_ij} - a_j}{x_{n_ij_0} - a_{j_0}} = \begin{cases} 0 & j \in J_1 \\ \alpha_j & j \in J_2 \\ \infty & j \in J_3 \end{cases} \tag{2.8}$$

where $\alpha_j \neq 0,\infty$ for all $j \in J_2$,

$$\sum_{i=1}^{\infty} (x_{n_ij_0} - a_{j_0})^{2r_0} = \infty\,, \tag{2.9}$$

$$(x_{n_ij} - a_j)^{r_0+1} = o((x_{n_ij_0} - a_{j_0})^{r_0}),\quad j \in J_3\,. \tag{2.10}$$

For $r = 1,2,\dots,r_0$

$$\varliminf_{i\to\infty} \left| \sum_{k=0}^{r-1} S_i(k,j_0) + S_i(r,j_0,T_{r3}) + \underset{\sim}{w}'\underset{\sim}{v}_{j_0}^{(r)}(\underset{\sim}{a}) \right| > 0 \tag{2.11}$$

for all $\underset{\sim}{w}$ with $\underset{\sim}{w}'\underset{\sim}{v}_{j_0}^{(r)}(\underset{\sim}{a}) \neq 0$ where $S_i(k,j_0)$, $S_i(r,j_0,T_{r3})$ and T_{r3} are defined in (2.2), (2.3) and (2.7). Then the vectors

$$v_{j_0}^{(r)}(a) = \sum_{t_r \in T_{r2}} \alpha_{j_1} \cdots \alpha_{j_r} \frac{\partial^r f(a)}{\partial x_{j_1} \cdots \partial x_{j_r}}, \qquad r = 1,2,\ldots,r_0$$

are consistent directions.

Proof: According to Theorem 1 of Wu (1980), we only need to prove that for any fixed $1 \le r \le r_0$

$$\sum_{i=1}^{\infty} [w' f(x_{n_i})]^2 = \infty \text{ for any } w \text{ with } w' v_{j_0}^{(r)}(a) \ne 0 .$$

Consider a Taylor series expansion of $f(x_{n_i})$ at a

$$f(x_{n_i}) = \sum_{k=0}^{r} [(x_{n_i} - a)\nabla]^k f(a)/k! + [(x_{n_i} - a)\nabla]^{r+1} f(\xi_i)/(r+1)!$$

where $\xi_i = x_{n_i} + \theta^{(i)}(x_{n_i} - a)$, $\theta^{(i)} = \mathrm{diag}(\theta_1^{(i)},\ldots,\theta_q^{(i)})$, $0 \le \theta_j^{(i)} \le 1$ for $1 \le j \le q$ and all i. Thus,

$$\sum_{i=1}^{\infty} [w' f(x_{n_i})]^2 = \sum_{i=1}^{\infty} (x_{n_i j_0} - a_{j_0})^{2r} \left[\sum_{k=0}^{r} \frac{w'[(x_{n_i} - a)\nabla]^k f(a)}{k!(x_{n_i j_0} - a_{j_0})^r} + \frac{w'[(x_{n_i} - a)\nabla]^{r+1} f(\xi_i)}{(r+1)!(x_{n_i j_0} - a_{j_0})^r} \right]^2$$

$$= \sum_{i=1}^{\infty} (x_{n_i j_0} - a_{j_0})^{2r} \Big[\sum_{k=0}^{r} S_i(k,j_0) + R_i(r+1,j_o) \Big]^2$$

$$= \sum_{i=1}^{\infty} (x_{n_i j_0} - a_{j_0})^{2r} S_{ir}^2 , \qquad (2.12)$$

where

$$R_i(r+1,j) = \frac{w'[(x_{n_i} - a)\nabla]^{r+1} f(\xi_i)}{(r+1)!(x_{n_i j_0} - a_{j_0})^r} .$$

By assumption (ii), there exists an $M > 0$, such that

$$\sup_{\underset{\sim}{x} \in N(\underset{\sim}{a},\delta)} \left| \frac{\partial^{r+1} f_t(\underset{\sim}{x})}{\partial x_{j_1} \cdots \partial x_{j_{r+1}}} \right| < M, \qquad t = 1,2,\ldots,p .$$

From Cauchy-Schwarz inequality, (2.8) and (2.10),

$$R_i(r+1,j_0) \le (\text{constant}) \| [(\underset{\sim}{x}_{n_i} - \underset{\sim}{a})\nabla]^{r+1} \underset{\sim}{f}(\underset{\sim}{\xi}_i) \| / |x_{n_i j_0} - a_{j_0}|^r \to 0 \quad \text{as} \quad i \to \infty .$$

We decompose $S_i(r,j_0)$ into two parts

$$S_i(r,j_0) = S_i(r,j_0,T_{r1} \cup T_{r2}) + S_i(r,j_0,T_{r3}) .$$

From (2.8), it is easy to see that

$$S_i(r,j_0,T_{r1} \cup T_{r2}) \to \underset{\sim}{w}' \sum_{\underset{\sim}{t}_r \in T_{r2}} \alpha_{j_1} \cdots \alpha_{j_r} \frac{\partial^r \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_{j_1} \cdots \partial x_{j_r}} = \underset{\sim}{w}' \underset{\sim}{v}_{j_0}^{(r)}(\underset{\sim}{a}) .$$

By assumption (2.11), there exist N_0 and $\eta > 0$ such that for $i > N_0$, $S_{ir}^2 > \eta$. Since (2.9) implies $\sum_{i=1}^{\infty} (x_{n_i j_0} - a_{j_0})^{2r} = \infty$ for any $r \le r_0$, from (2.12), the required result is proved. □

The linear space spanned by $\underset{\sim}{v}_{j_0}^{(r)}(\underset{\sim}{a})$, is independent of the choice of $j_0 \in J_2$. In fact, for any $j_1 \in J_2$, $j_1 \ne j_0$, let

$$\underset{\sim}{v}_{j_1}^{(r)}(\underset{\sim}{a}) = \sum_{\underset{\sim}{t}_r \in T_{r2}} \alpha_{j_1}^* \cdots \alpha_{j_r}^* \frac{\partial^r \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_{j_1} \cdots \partial x_{j_r}}$$

where

$$\alpha_j^* = \lim_{i \to \infty} (x_{n_i j} - a_j)/(x_{n_i j_1} - a_{j_1}), \qquad j \in J_2 .$$

It is easy to see that

$$\underset{\sim}{v}_{j_1}^{(r)}(\underset{\sim}{a}) = (\alpha_{j_0}^*)^r \underset{\sim}{v}_{j_0}^{(r)}(\underset{\sim}{a}) \quad \text{for} \quad 1 \le r \le r_0 .$$

Thus the two vectors span the same linear subspace.

When $J_1 = J_3 = \phi$, the empty set, all the components of $\underset{\sim}{x}_{n_i}$ converge at the same rate and Theorem 2 takes a simpler form.

<u>Corollary 1</u>. Suppose (i) and (ii) of Theorem 2 hold and for some $1 \le j_0 \le q$

$$\lim_{i\to\infty} \frac{x_{n_i j} - a_j}{x_{n_i j_0} - a_{j_0}} = \alpha_j \quad \text{for} \quad 1 \le j \le q \tag{2.13}$$

where $\alpha_j \ne 0,\infty$,

$$\sum_{i=1}^{\infty} (x_{n_i j_0} - a_{j_0})^{2r_0} = \infty . \tag{2.14}$$

Then

$$[\underset{\sim}{\alpha}\nabla]^r \underset{\sim}{f}(\underset{\sim}{a}) \text{ are consistent directions for } r = 1,2,\ldots,r_0$$

where

$$\underset{\sim}{\alpha} = (\alpha_1,\ldots,\alpha_q) \quad \text{and} \quad [\underset{\sim}{\alpha}\nabla]^r \underset{\sim}{f}(\underset{\sim}{a}) = \sum_{\underset{\sim}{t}_r \in T_r} \alpha_{j_1} \cdots \alpha_{j_r} \frac{\partial^r \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_{j_1} \cdots \partial x_{j_r}} .$$

Proof: It is sufficient to prove that (2.11) holds for the case under consideration. In fact, since $T_{r3} = \phi$, $S_i(r,j_0,T_{r3}) = 0$, and

$$S_i(k,j_0) = \frac{\underset{\sim}{w}'}{(x_{n_i j_0} - a_{j_0})^{r-k}} \sum_{\underset{\sim}{t}_k \in T_k} \frac{(x_{n_i j_1} - a_{j_1}) \cdots (x_{n_i j_k} - a_{j_k})}{(x_{n_i j_0} - a_{j_0}) \cdots (x_{n_i j_0} - a_{j_0})} \frac{\partial^k \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_{j_1} \cdots \partial x_{j_k}}$$

$$= \frac{\underset{\sim}{w}'}{(x_{n_i j_0} - a_{j_0})^{r-k}} S_i^*(k,j_0) \quad \text{for} \quad k = 0,1,\ldots,r .$$

From (2.13)

$$S_i^*(k,j_0) \to [\underset{\sim}{\alpha}\nabla]^k \underset{\sim}{f}(\underset{\sim}{a}), \qquad k = 1,2,\ldots,r ,$$

$$S_i^*(0,j_0) = \underset{\sim}{f}(\underset{\sim}{a}) .$$

Let k_0 be the first k with $\underset{\sim}{w}'[\underset{\sim}{\alpha}\nabla]^k \underset{\sim}{f}(\underset{\sim}{a}) \ne 0$. From $\underset{\sim}{w}'[\underset{\sim}{\alpha}\nabla]^r \underset{\sim}{f}(\underset{\sim}{a}) \ne 0$, we have $k_0 \le r$. Thus, the S_{ir}^2 in (2.12) are dominated by the leading term $S_i(k_0,j_0)$, which is bounded away from zero as $i \to \infty$. Therefore (2.11) is satisfied. □

Wu and Wang (1982) gave a more direct proof of Corollary 1.

If $J_3 = \phi$ and $J_2 = \{j_0\}$ in Theorem 2, we obtain immediately the following Corollary.

Corollary 2. Under conditions (i) and (ii) of Theorem 2 and the following conditions: for a fixed j_0

$$x_{n_i j} - a_j = o(x_{n_i j_0} - a_{j_0}) \quad \text{for all} \quad j \neq j_0 , \tag{2.15}$$

$$\sum_{i=1}^{\infty} (x_{n_i j_0} - a_{j_0})^{2r_0} = \infty , \tag{2.16}$$

$$\lim_{i\to\infty} \left| \sum_{k=0}^{r-1} S_i(k, j_0) + \underset{\sim}{w}' \frac{\partial^r \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_{j_0}^r} \right| > 0$$

$$\text{for all } \underset{\sim}{w} \text{ with } \underset{\sim}{w}' \frac{\partial^r \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_{j_0}^r} \neq 0 \text{ and } r = 1,2,\ldots,r_0 \tag{2.17}$$

where

$$\frac{\partial^r \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_{j_0}^r} = \left(\frac{\partial^r f_1(\underset{\sim}{a})}{\partial x_{j_0}^r}, \ldots, \frac{\partial^r f_p(\underset{\sim}{a})}{\partial x_{j_0}^r}\right)' .$$

Then

$$\frac{\partial^r \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_{j_0}^r} \text{ are consistent directions for } r = 1,2,\ldots,r_0 .$$

Similarly, for $J_1 = \phi$ and $J_2 = \{j_0\}$, analogous results readily obtain.

For multiple regression, we only need to consider a special case of Theorem 2, i.e. $r_0 = 1$. Two cases are considered below as corollaries.

Corollary 3. Suppose (i), (ii) and (2.8), (2.9), (2.10) of Theorem 2 for $r_0 = 1$ hold, and

$$\lim_{i\to\infty} \left| \frac{\underset{\sim}{w}' \underset{\sim}{f}(\underset{\sim}{a})}{x_{n_i j_0} - a_{j_0}} + \underset{\sim}{w}' \sum_{j\in J_3} \frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_j} \frac{(x_{n_i j} - a_j)}{(x_{n_i j_0} - a_{j_0})} + \underset{\sim}{w}' \sum_{j\in J_2} \alpha_j \frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_j} \right| > 0 \tag{2.18}$$

for all $\underset{\sim}{w}$ with

$$\underset{\sim}{w}' \sum_{j \in J_2} \alpha_j \frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_j} \neq 0 \ .$$

Then

$$\sum_{j \in J_2} \alpha_j \frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_j} \text{ is a consistent direction .}$$

From Corollaries 1 and 2 we observe that (2.11) holds automatically for $r_0 > 1$ only if J_1 and J_3 are both empty. Such is not the case for $r_0 = 1$. It is easy to see that if $J_3 = \phi$, (2.18) is automatically satisfied, and thus the following corollary.

Corollary 4. Suppose (i) and (ii) of Theorem 2 hold for $r_0 = 1$ and

(iii) there exists a partition J_1 and J_2 of the set $J = \{1,2,\ldots,q\}$ such that for a fixed $j_0 \in J_2$,

$$\lim_{i \to \infty} \frac{x_{n_i j} - a_j}{x_{n_i j_0} - a_{j_0}} = \begin{cases} 0 & j \in J_1 \\ \alpha_j & j \in J_2 \end{cases}$$

where $\alpha_j \neq 0,\infty$, and

$$\sum_{i=1}^{\infty} (x_{n_i j_0} - a_{j_0})^2 = \infty \ .$$

Then

$$\sum_{j \in J_2} \alpha_j \frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_j} \text{ is a consistent direction.}$$

The following theorem shows that by further partitioning J_3 into disjoint subsets more consistent directions will be obtained.

Theorem 3. Suppose (i) and (ii) of Theorem 2 hold, and there exists a partition $\{J_\ell\}_{\ell=1}^{h}$ of the set $J = \{1,2,\ldots,q\}$ satisfying the following conditions:

(iii) For a fixed $j_\ell \in J_\ell$, $\ell = 1,2,\ldots,h$

$$\lim_{i\to\infty} \frac{x_{n_i k} - a_k}{x_{n_i j_\ell} - a_{j_\ell}} = \alpha_{kj_\ell \ell} \quad \text{for all } k \in J_\ell$$

where $\alpha_{kj_\ell \ell} \neq 0,\infty$.

(iv) For any $j \in J_\ell$, $k \in J_{\ell+1}$, $\ell = 1,2,\ldots,h-1$

$$x_{n_i j} - a_j = o(x_{n_i k} - a_k) ,$$

(v) There exists an $h_0 \leq h$ such that for $h_0 \leq \ell \leq h$, there exists an integer r_ℓ s.t.

$$(x_{n_i k} - a_k)^{r_\ell+1} = o((x_{n_i j} - a_j)^{r_\ell}), \qquad k \in J_h,\ j \in J_\ell ,$$

$$\sum_{i=1}^{\infty} (x_{n_i j} - a_j)^{2r_\ell} = \infty, \qquad j \in J_\ell,\ \ \ell > h_0 .$$

(vi) For $h_0 \leq \ell \leq h$

$$\varliminf_{i\to\infty} \left| \sum_{k=0}^{r-1} S_i(k,j_\ell) + S_i(r,j_\ell,T_\ell^*) + \underset{\sim}{w}'\underset{\sim}{v}_{j_\ell}^{(r)}(\underset{\sim}{a}) \right| > 0$$

for $\underset{\sim}{w}$ with $\underset{\sim}{w}'\underset{\sim}{v}_{j_\ell}^{(r)}(\underset{\sim}{a}) \neq 0, \quad r = 1,2,\ldots,r_\ell^*$

where $T_\ell^* = \{(j_1,\ldots,j_r) : (j_1,\ldots,j_r) \in T_r$ for at least one $j_t \in \bigcup_{k=\ell+1}^{h} J_k\}$,
$r_\ell^* = \min\{r_t : t > \ell\}$.

Then the vectors

$$\underset{\sim}{v}_{j_\ell}^{(r)}(\underset{\sim}{a}) = \sum_{j_1,\ldots,j_r \in J_\ell} \alpha_{j_1 j_\ell \ell} \cdots \alpha_{j_r j_\ell \ell} \frac{\partial^r \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_{j_1} \cdots \partial x_{j_r}} ,$$

$$\ell = h_0, h_0+1,\ldots,h, \quad r = 1,2,\ldots,r_\ell^*$$

are consistent directions.

Proof: For any fixed J_ℓ, $h_0 \le \ell \le h-1$, $j_0 \in J_\ell$, we regard J_ℓ, j_ℓ, $\bigcup_{k=1}^{\ell-1} J_k$ and $\bigcup_{k=\ell+1}^{h} J_k$ as J_2, j_0, J_1 and J_3, respectively, in Theorem 2. From Theorem 2 we conclude that $\underset{\sim}{v}_{j_\ell}^{(r)}(\underset{\sim}{a})$ are consistent directions for $\ell = h_0, h_0+1, \ldots, h-1$, and $r = 1,2,\ldots,r_\ell^*$. That $\underset{\sim}{v}_{j_h}^{(r)}(\underset{\sim}{a})$ are consistent directions for $r = 1,2,\ldots,r_0$ follows from Theorem 2 with $J_3 = \phi$. □

By combining the results in Theorems 1 and 3 for each limit point of the design sequence and using Theorem 2 of Wu (1980), we obtain the following main theorem.

Define $L\{\underset{\sim}{a}_1,\ldots,\underset{\sim}{a}_t\}$ to be the subspace spanned by vectors $\underset{\sim}{a}_1,\ldots,\underset{\sim}{a}_t$.

<u>Theorem 4</u>. Suppose $\underset{\sim}{a}_j$, $j = 1,\ldots,k$, are k distinct limit points of $\{\underset{\sim}{x}_i\}_{i=1}^{\infty}$ and $\underset{\sim}{v}_\ell(\underset{\sim}{a}_j)$, $j = 1,\ldots,k$, $\ell = 1,\ldots,t_j$, are the consistent directions obtained from Theorem 3, then

$$S(\underset{\sim}{f}) = \sum_{j=1}^{k} A_j(\underset{\sim}{f}) \oplus B_{k+1}(\underset{\sim}{f})$$

where

$$A_j(\underset{\sim}{f}) = L\{\underset{\sim}{f}(\underset{\sim}{a}_j),\ \underset{\sim}{v}_\ell(\underset{\sim}{a}_j),\ \ell = 1,\ldots,t_j\}, \qquad j = 1,\ldots,k\ ,$$

$$B_{k+1}(\underset{\sim}{f}) = \{\underset{\sim}{u} \in [\sum_{j=1}^{k} A_j(\underset{\sim}{f})]^{\perp} : \sum_{i=1}^{\infty} [\underset{\sim}{w}'\underset{\sim}{f}(\underset{\sim}{x}_i)]^2 = \infty$$

$$\text{for any } \underset{\sim}{w} \in [\sum_{j=1}^{k} A_j(\underset{\sim}{f})]^{\perp} \text{ and } \underset{\sim}{w}'\underset{\sim}{u} \ne 0\}\ .$$

There is no loss of generality in assuming finite k in Theorem 4 as was noted in Wu and Wang (1982).

3. Examples

In this section the general results developed heretofore will be applied to some regression models. These examples show that our conditions are easier to verify than the more typical condition $\underset{\sim}{b}'(X_n'X_n)^{-1}\underset{\sim}{b} \to 0$.

(i) Multiple regression

$$\underset{\sim}{y} = \underset{\sim}{\theta}'\underset{\sim}{f}(\underset{\sim}{x}) + \varepsilon \tag{3.1}$$

where $\underset{\sim}{\theta}' = (\theta_0,\theta_1,\ldots,\theta_{q-1})$, $\underset{\sim}{x}' = (1,x_1,\ldots,x_{q-1})$, $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{x}$.

For the design sequence $\underset{\sim}{x}_i' = (x_{i0},x_{i1},\ldots,x_{iq-1})$, $x_{i0} = 1$, $i = 1,2,\ldots,\underset{\sim}{x}_i \to \underset{\sim}{a} = (a_0,a_1,\ldots,a_{q-1})'$, $a_0 = 1$.

(a) If $x_{ij} = a_j + i^{-1/2}$, $j \geq 1$, Corollary 4 with $\alpha_j = 1$, $j = 1,2,\ldots,q-1$, $r_0 = 1$, $J_1 = \{0\}$, $J_2 = \{1,\ldots,q-1\}$ applies. From Theorem 1 and Corollary 4, $\underset{\sim}{f}(\underset{\sim}{a}) = \underset{\sim}{a}$, $\sum_{j=1}^{q-1} \frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_j} = (0,1,\ldots,1)'$ are consistent directions.

(b) If $x_{i1} = a_1 + i^{-1/2}$, $x_{ij} = a_j + i^{-1/3}$, $j \geq 2$. Partition $J = \{0,1,2,\ldots,q-1\}$ into $J_1 = \{0\}$, $J_2 = \{1\}$, $J_3 = \{2,\ldots,q-1\}$. Therefore, $\underset{\sim}{a}$, $\frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_1} = (0,1,0,\ldots,0)'$ and $\sum_{j=2}^{q-1} \frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_j} = (0,0,1,1,\ldots,1)'$ are consistent directions from Theorem 1, Corollaries 3 and 4, respectively.

(c) If $x_{ij} = a_j + i^{-(j+1)^{-1}}$, $j \geq 1$. It is easy to verify that $J_\ell = \{\ell\}$, $\ell = 0,1,2,\ldots,q-1$, $h_o = [q/2]$, where $[x]$ is the largest integer less than or equal to x, J_ℓ and h_o are defined in Theorem 3. Therefore $\frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_j} = \underset{\sim}{e}_j = (0,\ldots,0,1,0,\ldots,0)'$, $j = [q/2],\ldots,q$, and $\underset{\sim}{a}$ are consistent directions.

(ii) Multiple polynomial regression

$$\underset{\sim}{y} = \underset{\sim}{\theta}'\underset{\sim}{f}(\underset{\sim}{x}) + \varepsilon \tag{3.2}$$

where $f_j(\underset{\sim}{x})$, $j = 1,\ldots,p$, are monomials in $\underset{\sim}{x}$ of degree less than or equal to d. If $f_i(\underset{\sim}{x}) \neq f_j(\underset{\sim}{x})$ for $i \neq j$, then $p = (d+q)!/(d!q!)$.

We will describe in more detail the general results to be obtained for the following quadratic polynomial regression model in two variables:

$$y = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1 x_2 + \theta_4 x_1^2 + \theta_5 x_2^2 + \varepsilon \tag{3.3}$$

and limit point $\underset{\sim}{a} = (a_1,a_2)'$.

(a) If $x_{ij} = a_j + i^{-1/4}$, $1 \le j \le q$, Corollary 1 with $\alpha_j = 1$, $j = 1,\dots,q$ and $r_0 = 2$ applies. Thus $\underset{\sim}{f}(\underset{\sim}{a})$, $\sum_{j=1}^{q} \frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_j}$ and $\sum_{i,j=1}^{q} \frac{\partial^2 \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_i \partial x_j}$ are consistent directions for model (3.2). In particular, for model (3.3), the three consistent directions are

$$\underset{\sim}{f}(\underset{\sim}{a}) = (1,a_1,a_2,a_1a_2,a_1^2,a_2^2)' ,$$

$$\frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_1} + \frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_2} = (0,1,1,a_1 + a_2, 2a_1, 2a_2)' ,$$

$$\sum_{i,j=1}^{2} \frac{\partial^2 \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_i \partial x_j} = (0,0,0,2,2,2)' .$$

(b) If $x_{ij} = a_j + i^{-1/2}$ for $1 \le j \le j_o$ and $= a_j + i^{-1/3}$ for $j_o < j \le q$, then $J_1 = \{1,2,\dots,j_o\}$, $J_2 = \{j_o + 1,\dots,q\}$, $r_1 = r_2 = 1$ in Theorem 3. Therefore $\underset{\sim}{f}(\underset{\sim}{a})$, $\sum_{j=1}^{j_o} \frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_j}$ and $\sum_{j=j_o+1}^{q} \frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_j}$ are consistent directions. For model (3.3) and $j_0 = 1$, the last two consistent directions are

$$\frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_1} = (0,1,0,a_2, 2a_1,0)' ,$$

$$\frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_2} = (0,0,1,a_1,0, 2a_2)' .$$

(c) If $x_{ij} = a_j + i^{-1/(j+1)}$, $j = 1,2,\dots,q$, it is easy to verify that $J_\ell = \{\ell\}$, $\ell = 1,2,\dots,q$, $h_o = [\frac{q+1}{2}]$, $r_\ell = \min\{[\frac{\ell+1}{2}], [\frac{\ell+1}{q-\ell}]\}$ and $r_\ell^* = 1$ for all $\ell > h_o$ where J_ℓ, h_o, r_ℓ and r_ℓ^* are defined in Theorem 3. Thus we know that $\underset{\sim}{f}(\underset{\sim}{a})$, $\frac{\partial \underset{\sim}{f}(\underset{\sim}{a})}{\partial x_j}$, $j = [(q + 1)/2],\dots,q$, are consistent directions.

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The consistent directions of the least squares estimators in a linear model are defined to be the linear combinations of parameter estimates that are asymptotically consistent. When the design variable is univariate and the regression function is smooth, consistent directions are characterized in previous papers (Wu, 1980; Wu and Wang, 1982) in terms of the convergence rates of the design sequence to its limit points. Extensions of these results to multivariate design variables are considered in the present paper.